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Damping of the wave packet motion in a general time-dependent quadratic field

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Abstract. We provide a framework for the study of a quantal time-dependent oscillator in the presence of a loss mechanism. Previous approaches to partial aspects of the problem are analysed and cast into a unified global picture. Several alternative descriptions of the situation are analysed and it is shown that the proper generalisation of the Hamiltonian formulation by Kostin is adequate for the case under consideration. Applications to a number of problems in which the mass of the oscillator is a given function of time are presented, including cases in which the mass becomes infinite.

1. Introduction

The problem of quantising the damped motion of a particle in a quadratic field has received considerable attention (for a review see Hasse 1975). These previous works usually deal with an oscillator with constant mass and stiffness placed in the presence of a dissipative force $F = -\gamma\dot{X}$, γ also being a constant. In addition, there exists a substantial body of work concerning the study of a classical, undamped harmonic oscillator with arbitrary time dependence in its parameters (Lewis 1967, Symon 1970, Howard 1970).

The present paper aims at unifying these views in order to provide a treatment of a quantal oscillator in the presence of a dissipation mechanism, in the most general situation in which the mass, the quadratic field and the damping are arbitrary functions of time. To achieve this goal, we will generalise the classical problem, allowing the time-dependent oscillator to be damped (see § 2). In § 3, we take advantage of standard quantisation rules and derive equations of motion for both the first and second moments of the wavefunction. In § 4, we introduce the proper generalisation of already existing quantal descriptions of damping that do not possess a classical equivalent (Hasse 1975). An analysis of the approaches presented in §§ 3 and 4 allows us to choose one satisfactory representation of the quantal problem. We limit ourselves to the study of Gaussian wave packets; this restriction is not a very dramatic one, since a number of applications can be derived on that assumption. Some typical examples will be presented and discussed in § 5.

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2. The classical problem

In this section we will discuss the classical description of the motion of a damped oscillator. The firm basis for this study is, of course, Newton's equation:

$$\dot{P} = -\gamma(t)P - m(t)\Omega^2(t)X. \tag{1}$$

The kinetic momentum is defined as

$$P = m(t)\dot{X}. \tag{2}$$

The dotted variables are time derivatives and the friction parameter γ, the mass m and the frequency Ω may depend on time.

Although equation (1) cannot be derived from a Lagrangian containing a conservative potential, we can write a time-dependent Lagrangian that leads to the correct equations of motion, namely

$$L(X, \dot{X}, t) = \tfrac{1}{2}m_0 \exp(f(t))[\dot{X}^2 - \Omega^2(t)X^2]. \tag{3}$$

This is a generalisation of the Lagrangian proposed by Kanaĭ (1948). Here $f(t)$ contains the time dependence of both mass and friction; we require only that $f(t)$ and $\Omega(t)$ are differentiable functions of time. The problem originally discussed by Kanaĭ is recovered when $f(t) = \gamma t$ with constant γ.

The canonical momentum is defined as

$$\bar{P} = \partial L/\partial \dot{X} = m_0 \dot{X} \exp(f(t)). \tag{4}$$

It differs from the kinetic momentum P given by equation (2); their relationship is

$$\bar{P} = (m_0/m(t)) \exp(f(t))P. \tag{5}$$

Accordingly the Hamiltonian associated with the Lagrangian (3) reads

$$H = (\bar{P}^2/2m_0) \exp(-f(t)) + (m_0/2)\Omega^2(t)X^2 \exp(f(t)). \tag{6}$$

Notice that in general we cannot identify H with the energy E of the oscillator, since

$$E = T + V = (P^2/2m) + \tfrac{1}{2}m\Omega^2X^2.$$

We have $E \equiv H$ only when P and $\bar{P}$ coincide; this happens in the absence of dissipation for any time dependence of m and Ω.

Hamilton's equations of motion are

$$\dot{X} = \bar{P}/m_0 \exp(-f(t)), \tag{7a}$$

$$\dot{\bar{P}} = -m_0\Omega^2X \exp(f(t)). \tag{7b}$$

If we introduce the kinetic momentum (see equation (2)), we recognise Newton's equation,

$$\ddot{X} + \dot{f}\dot{X} + \Omega^2X = 0.$$

In addition, equations (7a) and (7b) lead to the rate of energy change,

$$\dot{E} = (\dot{m}/m - 2\dot{f})P^2/2m + \tfrac{1}{2}(\dot{m}/m + 2\dot{\Omega}/\Omega)m\Omega^2X^2. \tag{8}$$

If we consider a damped harmonic oscillator with constant mass and frequency, $\dot{E}$ takes the well-known form

$$\dot{E} = -\gamma(P^2/m).$$

To introduce a time dependence for the mass, we can write

$$f(t) = \log m/m_0 + \gamma t$$

and

$$\dot{E} = -(\gamma + \dot{m}/2m)P^2/m.$$

We can see that the variation of mass acts like an extra damping term when $\dot{m}$ is positive.

3. The quantal problem

The problem of the quantisation of a damped harmonic oscillator has been analysed by several authors (see, for example, Hasse (1975)) in the case of constant mass and stiffness. Two lines are usually followed: (i) the straightforward quantisation of the classical Hamiltonian enforcing the principle of correspondence, and (ii) the construction of an *ad hoc* Hamiltonian that does not possess a classical analogue.

Both approaches are legitimate if

(i) the expectation values

$$x = \langle \hat{x} \rangle \qquad \text{and} \qquad \bar{p} = \langle \hat{\bar{p}} \rangle \tag{9}$$

(where $\hat{x}$ and $\hat{\bar{p}}$ are respectively the quantal operators associated with the canonical variables) satisfy the classical equations of motion, since these are linear;

(ii) the uncertainties in coordinate and kinetic momentum fulfil the Heisenberg relation. The quantisation procedure only guarantees that the relation is preserved for the canonical variables and particular attention must be paid to the formalisms in which canonical and kinetic momenta differ. Additional restrictions may be provided, depending on the particular problem we want to analyse.

The quantal operators $\hat{x}$ and $\hat{\bar{p}}$ display fluctuations that do not have a classical equivalence. Following Hasse (1978), we define χ, ϕ, the fluctuations of $\hat{x}$ and $\hat{\bar{p}}$, respectively, and the correlation σ, as

$$\chi = \langle \hat{x}^2 \rangle - x^2, \tag{10a}$$

$$\phi = \langle \hat{\bar{p}}^2 \rangle - \bar{p}^2, \tag{10b}$$

$$\sigma = \tfrac{1}{2}\langle \hat{x}\hat{\bar{p}} + \hat{\bar{p}}\hat{x} \rangle - x\bar{p}. \tag{10c}$$

As $\hat{x}$ and $\hat{\bar{p}}$ are canonical conjugates, they satisfy the Heisenberg uncertainty principle

$$\chi\phi \geqslant \hbar^2/4. \tag{11}$$

Most physical situations we are interested in are expressed through Gaussian wave packets, for which the inequality (11) becomes a strict equality binding the three fluctuations:

$$\chi\phi = \sigma^2 + \hbar^2/4. \tag{12}$$

The time evolution of the first (equation (9)) and second (equation (10)) moments of the wave packets can be derived using the definition of the total derivative of a given operator $\hat{A}$:

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \frac{\partial \hat{A}}{\partial t} + \frac{1}{\mathrm{i}\hbar}[\hat{A}, \hat{H}]. \tag{13}$$

The first moments x and $\bar{p}$ are fixed by Ehrenfest's limit, and the second moments are solutions of a set of three coupled first-order differential equations. In most quantisation procedures, as we will see later, this system may be decoupled and provides a nonlinear differential equation for χ, namely

$$\chi\ddot{\chi} - \tfrac{1}{2}(\dot{\chi})^2 + g(t)\chi\dot{\chi} + 2h(t)\chi^2 = \hbar^2/2k(t).$$

Here, $g(t)$, $h(t)$ and $k(t)$ are well-defined functions of time whose explicit form depends on the particular time-variation of the damping and the harmonic oscillator parameters.

This equation takes a simpler form, if we introduce the width u of the wave packet:

$$u(t) = (2/\hbar)^{1/2}\chi^{1/2}.$$

We obtain

$$\ddot{u} + g(t)\dot{u} + h(t)u = [k(t)u^3]^{-1}. \tag{14a}$$

This is an equation describing a forced and damped oscillator. Although in the general case we do not dispose of analytical solutions, they are available in a few simple situations that we explore later.

It is interesting to notice that if we introduce the reduced width $W(t)$ through the definition

$$u(t) = W(t)\exp\left(-\tfrac{1}{2}\int g(t)\,\mathrm{d}t\right),$$

we can rewrite equation (14a) in a compact form

$$\ddot{W} + \omega^2 W = d(t)W^{-3} \tag{14b}$$

that proves to be useful when comparing different formalisms. The reduced frequency $\omega(t)$ and the function $d(t)$ can be shown to be

$$\omega^2 = h - \tfrac{1}{4}g^2 - \tfrac{1}{2}\dot{g},$$

$$d = k^{-1}\exp\left(2\int g(t)\,\mathrm{d}t\right).$$

Equations (14a) and (14b) are generalisations of the auxiliary equation introduced by Lewis (1967) in his search for the invariant of the classical time-dependent oscillator. It can be proved that a quantal invariant operator may always be written in terms of the fluctuations in the case of the undamped oscillator (see Hernandez and Remaud 1980). In the presence of damping, such a quantal invariant can be found only in some specific cases (Remaud and Hernandez 1979).

4. Quantal Hamiltonians

In this section we will present in a unified description the various approaches used in the literature to study damped oscillators and we shall extend them to the general time-dependent oscillator. Since all these approaches have been chosen to provide the exact equations of motion for x and p, we will concentrate on the properties of the second moments.

4.1. Presentation of the various Hamiltonians

4.1.1. General time-dependent quantal Hamiltonian. Straightforward quantisation of equation (6) gives

$$\hat{H} = -(\hbar^2/2m_0)\frac{\partial^2}{\partial x^2}(\exp(-f)) + (m_0/2)\Omega^2(t)\exp(f)\hat{x}^2. \tag{15}$$

In order to write the Hamiltonian $\hat{H}$, we have quantised the canonical momentum $\bar{P}$, whose relationship with the kinetic momentum P is given by equation (5). This feature induces a difference between the fluctuation ϕ as defined in equation (10*b*) and the uncertainty in the kinetic momentum Δp; we have

$$(\Delta p)^2 = (m/m_0)^2 \exp(-2f(t))\phi.$$

Thus the uncertainty product of position and kinetic momentum follows the law

$$\Delta x \Delta p \geqslant (\hbar/2)m/m_0 \exp(-f(t)). \tag{16}$$

It is interesting to notice that if we define the energy operator $\hat{E}$ as

$$\hat{E} = (m/m_0)\hat{H}\exp(-f(t))$$

the mean value $\langle\hat{E}\rangle$ satisfies exactly the equation (8) for the classical rate of change (see Hasse (1975) for the discussion of the oscillator with constant mass and stiffness).

Applying equation (13) to the quantal operators in equations (10) we find the following set of coupled first-order differential equations:

$$\dot{\chi} = 2(m_0\exp(f))^{-1}\sigma,$$
$$\dot{\phi} = -2(m_0\exp(f))\Omega^2\sigma,$$
$$\dot{\sigma} = -(m_0\exp(f))\Omega^2\chi + (m_0\exp(f))^{-1}\phi.$$

We can easily check that the condition (12) defining a Gaussian wave packet is preserved by the above equations of motion.

In table 1, we list the actual form for the functions $g(t)$, $h(t)$, $k(t)$, $\omega(t)$ and $d(t)$ as defined in the preceding section. In this particular case, it can be shown that the motion of ϕ can be decoupled; and in the same way as for χ, we can define a width $v(t)$ for the momentum distribution:

$$v(t) = (2/\hbar)^{1/2}\phi^{1/2}.$$

Table 1. Comparison among the functions $g(t)$, $h(t)$, $k(t)$, $\omega^2(t)$, and $d(t)$ appearing in § 3 of the text for the various Hamiltonians analysed in the paper. Notice that if $f(t) \equiv \ln(m/m_0) + \int\gamma(t)\,dt$, the reduced frequencies ω^2 for the GTD and Hasse (i.e. $\gamma' = \gamma/2$) Hamiltonians are identical.

	General time-dependent (GTD) Hamiltonian	Hasse Hamiltonians	Kostin Hamiltonian
$g(t)$	$\dot{f}$	$\dot{m}/m$	$\gamma + \dot{m}/m$
$h(t)$	Ω^2	$\Omega^2 - \gamma'^2 - \gamma'\dot{m}/m - \dot{\gamma}'$	Ω^2
$k(t)$	$m_0^2\exp(2f)$	m^2	m^2
$\omega^2(t)$	$\Omega^2 - \frac{1}{4}(\dot{f})^2 - \frac{1}{2}\ddot{f}$	$h(t) + \left(\frac{\dot{m}}{2m}\right)^2 - \frac{\ddot{m}}{2m}$	$h(t) + \left(\frac{\dot{m}}{2m}\right)^2 - \frac{\ddot{m}}{2m} - \frac{1}{2}\gamma\frac{\dot{m}}{m} - \frac{\gamma^2}{4} - \frac{\dot{\gamma}}{2}$
$d(t)$	1	1	$\exp(2\gamma t)$

$v(t)$ is the solution of a second-order differential equation

$$\ddot{v} - (\dot{f} + 2\dot{\Omega}/\Omega)\dot{v} + \Omega^2 v = m_0^2 \Omega^2 \exp(2f)/v^3. \tag{17}$$

4.1.2. Hasse's Hamiltonians. Hasse (1978) has shown that the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m(t)}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}m(t)\Omega^2(t)\hat{x}^2 + \gamma(t)(\hat{x} - x)[\epsilon\hat{p} + (1-\epsilon)p] - \frac{i\hbar}{2}\epsilon\gamma(t) \tag{18}$$

synthesises the several nonlinear quantal frictional potentials presented in the literature. They differ only by the value of the parameter ϵ: Albrecht (1975) has studied the case $\epsilon = 0$, Süssmann (1973 Los Alamos seminar talk, unpublished) the case $\epsilon = 1$ and Hasse (1975) $\epsilon = \pm\frac{1}{2}$.

We can assume any time-dependence for m, γ and Ω; the equations of motion for the fluctuations are the same as for the time-independent oscillator (Hasse 1978):

$$\dot{\chi} = 2\gamma'\chi + 2\sigma/m, \qquad \dot{\phi} = -2\gamma'\phi - 2m\Omega^2\sigma,$$
$$\dot{\sigma} = -m\Omega^2\chi + \phi/m, \qquad \text{with } \gamma' = \epsilon\gamma.$$

These expressions are consistent with the conservation of the Gaussian shape of the wave packet (see equation (12)). Under this requirement, we can extract closed-form equations for the second moments. The equation for χ is written as described in § 3 (see equations (14*a*) and (14*b*)) and the results are displayed in table 1. Similar expressions can be found for ϕ and the momentum width $v(t)$ as follows:

$$\ddot{v} - (\dot{m}/m + 2\dot{\Omega}/\Omega)\dot{v} + [\Omega^2 - \gamma'^2 - \gamma'(\dot{m}/m + 2\dot{\Omega}/\Omega) + \dot{\gamma}']v = m^2\Omega^2/v^3,$$

which is to be compared with equation (17).

The frictional term in the Hamiltonian (18) has been devised to yield the correct Ehrenfest limit for the centre of the wave packet. It does not possess a classical analogue. In fact, the expectation value of the operator $\hat{V}_{\rm fr}$,

$$\hat{V}_{\rm fr} = \hat{H} - \hat{T} - \hat{V},$$

is

$$\langle\hat{V}\rangle_{\rm fr} = \gamma'\sigma.$$

In general cases, the energy differs from the expectation value of the Hamiltonian.

4.1.3. Kostin's Hamiltonian. It is also well-known that a former representation for friction on a quantal particle is due to Kostin (1975). It has been investigated too by Kan and Griffin (1974) in their fluid mechanical interpretation of the Schrödinger equation. The generalisation for the general time-dependent oscillator reads:

$$\hat{H} = -\frac{\hbar^2}{2m(t)}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}m(t)\Omega^2(t)\hat{x}^2 - \frac{i\hbar}{2}\gamma\left(\ln\frac{\psi}{\psi^*} - \left\langle\ln\frac{\psi}{\psi^*}\right\rangle\right) \tag{19}$$

where ψ and ψ^* are the wavefunction and its complex conjugate, respectively. This time-dependent Hamiltonian has a solution that describes the motion of a Gaussian wave packet (Hasse 1975):

$$\psi(x,t) = (2\pi)^{-1/4}\left(\frac{1}{\alpha} + \frac{1}{\alpha^*}\right)^{1/4}\exp\left\{-\frac{(\hat{x}-x)^2}{2\alpha} + \frac{i}{\hbar}\left[p(\hat{x}-x) + \int L\,dt - \theta\right]\right\}.$$

Here α is the complex time-dependent width, whose relationship with the real fluctuation χ is (Hasse 1978)

$$\tfrac{1}{2}\chi^{-1} = \mathrm{Re}(\alpha^{-1}).$$

In addition, L is equivalent to the classical Lagrangian:

$$L(x, p, t) = p^2/2m - \tfrac{1}{2}m\Omega^2 x^2$$

and θ is a real phase factor.

With the above wave packet, the frictional term in Kostin's Hamiltonian takes the form

$$\hat{V}_{\mathrm{fr}} = \gamma p(\hat{x} - x) + \tfrac{1}{4}i\hbar\gamma[(\hat{x} - x)^2 - \chi](1/\alpha - 1/\alpha^*).$$

It is easy to verify that $\langle \hat{V}_{\mathrm{fr}} \rangle$ is exactly zero, and then the expectation value of the energy is identical to the expectation value of $\hat{H}$.

The equation of motion for the second moments are, in this case,

$$\dot{\chi} = 2\sigma/m(t), \tag{20a}$$

$$\dot{\phi} = -2m(t)\Omega^2(t)\sigma - 2\gamma(t)\phi + \tfrac{1}{2}\hbar^2\gamma(t)/\chi, \tag{20b}$$

$$\dot{\sigma} = -m(t)\Omega^2(t)\chi + \phi/m(t) - \gamma(t)\sigma. \tag{20c}$$

As in the previous examples, the motion of χ and ϕ can be decoupled (see table 1).

4.2. Discussion

The aim of this section is to provide arguments for the choice of the best description of damping on a time-dependent oscillator.

(a) The general time-dependent Hamiltonian presents a serious shortcoming in the presence of damping. If we examine the equation (16) we realise that for sufficiently long time the uncertainty product $\Delta x \Delta p$ can become smaller than $\hbar/2$. This is a common feature to all descriptions that use different operators to represent classical and canonical moments. Senitzky (1960) has shown that this effect is due to the neglect of the fluctuations in the loss mechanism itself. This limitation hampers the use of the Hamiltonian (15) in the quantal study of the damped harmonic oscillators.

(b) The Hamiltonians of §§ 4.1.2 and 4.1.3 do not violate the uncertainty principle, and they yield the correct equations of motion for x and p through their nonlinear frictional part. The properties of nonlinear quantum mechanics have been little studied (see Immele *et al* (1975) for elements of discussion in the case of the free damped wave packet); nonlinear Hamiltonians do not allow the superposition of the wavefunctions and then forbid any perturbative approach of their solutions. This difficulty may be overcome each time one deals with well-defined wave packets that are exact solutions of the time-dependent Schrödinger equation.

Hamiltonians of §§ 4.1.2 and 4.1.3 do not give the same set of equations for the time dependence of fluctuations, and we need to examine their solutions for the damped harmonic oscillator with constant parameters, since we feel that a satisfactory description of damping should account for the decay to the ground state of any wave packet containing excited states.

From table 1, we see that with Hasse's Hamiltonians, the equation for the reduced width $W(t)$ takes the following form:

$$\ddot{W} + \omega^2 W = W^{-3} \qquad \text{where } \omega^2 = \Omega^2 - \gamma'^2.$$

The most general solution can be written as

$$W^2(t) = \omega^{-1}[(1+A^2+B^2)^{1/2} + A\cos 2\omega t + B\sin 2\omega t].$$

If we take the initial conditions

$$W(t=0) = W_0 \qquad \text{and} \qquad \dot{W}(t=0) = 0,$$

we obtain†

$$W^2(t) = W_0^2 + [W_0^2 - (\omega^2 W_0^2)^{-1}]\sin^2 \omega t. \tag{21}$$

The constant solution corresponds to a fluctuation $\chi = \hbar/2m\omega$. In addition, if W_0 is different from $\omega^{-1/2}$, we see from equation (21) that W will oscillate without decaying to the ground state as we should expect. The wave packet always keeps the memory of its initial width in the amplitude of its oscillations; and it is specially puzzling that even if W_0 takes the ground-state value $\Omega^{-1/2}$, the wave packet width will undergo infinite oscillations whose amplitude depends on the damping coefficient γ'.

These observations lead us to the conclusion that Hasse's Hamiltonians are not completely adequate for the treatment of the damped harmonic oscillator.

(c) In order to initiate the same study with Kostin's Hamiltonian, we recall the equation for the reduced width for the time independent oscillator (see table 1):

$$\ddot{W} + \omega^2 W = W^{-3}\exp(2\gamma t) \qquad \text{where } \omega^2 = \Omega^2 - \gamma^2/4.$$

In this case, it is not easy to write down $W(t)$ in the form of equation (21). Instead, we will find it useful to discuss the corresponding equation for the width u:

$$\ddot{u} + \gamma\dot{u} + \Omega^2 u = m^{-2}u^{-3}. \tag{22}$$

We can easily realise that there is a constant solution

$$u = (m\Omega)^{-1/2}. \tag{23}$$

The corresponding fluctuation χ turns out to be

$$\chi = \hbar/(2m\Omega),$$

and this is the actual dispersion of the ground state. It is illustrative to look for a solution of equation (22) that differs only slightly from the constant value (23).

Let us write $u(t)$ as

$$u(t) = (m\Omega)^{-1/2} + \delta(t), \tag{24}$$

with $\delta(t)$ such that $|\delta(t)| \ll (m\Omega)^{-1/2}$ for all t.

Straightforward substitution of equation (24) into equation (22) yields $\ddot{\delta} + \gamma\dot{\delta} + (2\Omega)^2\delta \simeq 0$.

It means that

$$\delta = \delta_0 \exp(-\gamma t/2)\sin(2\Omega t + \phi_0).$$

This shows that any state that differs only slightly from the ground state will decay to it.

We observe from equation (22) that there do not exist steady solutions with large deviations from the constant value (23), since in that case, the right-hand side becomes negligible and u behaves like the position of a damped oscillator.

† The authors thank Dr Hasse for pointing out a mistake in the first version of the manuscript.

This is illustrated in figure 1. The time evolution of χ and the total energy are displayed for two different initial conditions, namely, $\chi_0 = 0{\cdot}2$ and $\chi_0 = 4$, in units of the ground-state fluctuation. We observe that χ reaches the ground-state value in less than three periods. The same is true for x, although it should be remarked that the number of oscillations undergone by the fluctuation is twice that of the coordinate. It is also interesting to look at the evolution of the energy. The slight initial difference between the two curves can be traced back to the contribution of the fluctuations χ and ϕ to the potential and kinetic terms, respectively.

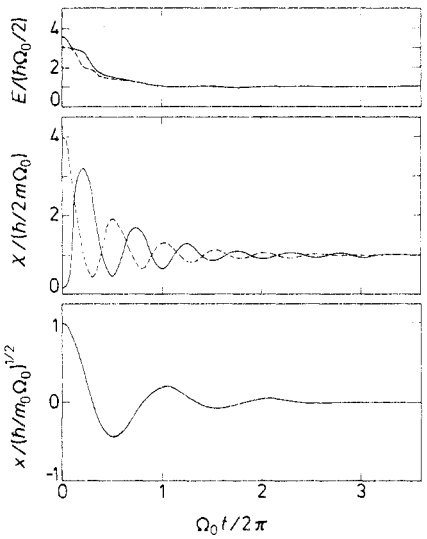

Figure 1. Time evolutions of energy, fluctuation and position of a Gaussian wave packet in units of $\hbar\Omega_0/2$, $\hbar/(2m_0\Omega_0)$ and $\hbar^{1/2}/(m_0\Omega_0)^{1/2}$ respectively. The time unit is the natural period. These calculations correspond to Kostin's potential with constant mass and stiffness; the damping parameter is 0·5 in units of the inverse period. The full lines correspond to an initial fluctuation $\chi_0 = 0{\cdot}2$, the dashed lines are for an initial $\chi_0 = 4$. The initial displacement x_0 is always equal to 1.

In view of these considerations and since the introduction of the time dependence in the oscillator parameters does not change the form of the equations, we can state that Kostin's Hamiltonian description is adequate to face the study of a general time-dependent Gaussian wave packet. In the following section, we will present several applications of this formalism.

5. Applications

In the next two sections we are going to study some specific interplays between mass variation and damping, that might be connected with a number of physical situations. In § 5.1 we will present typical results for systems initially in the ground state, that undergo a displacement and different mass variation laws. In § 5.2 we deal with a specific example inspired by situations appearing in heavy-ion physics.

5.1. Departures from the ground state

The system we will consider is a displaced ground-state wave packet, that presents at $t = 0^-$ the following characteristics:
$m_0 = 1$, $c_0 = 1$, $\chi_0 = \hbar/2$, $\phi_0 = \hbar/2$, $\sigma_0 = 0$ and $x_0 = 1$, $p_0 = 0$; we consider that at $t = 0$ a mass variation law $m(t)$ is superimposed on m_0.

5.1.1. Exponentially increasing mass. We choose the following variation law for $m(t)$:

$$m = 1; \qquad t \leqslant 0,$$
$$m = e^{\lambda t}; \qquad t \geqslant 0, \lambda \geqslant 0.$$

In this example, the mass presents a singularity at infinite time. Small values of λ provide us with a way of studying adiabatic behaviours. As $\dot{m}/m = \lambda$, the functions $g(t)$ and $\omega(t)$ (see table 1) take the following simple form:

$$g(t) = \gamma + \lambda,$$
$$\omega^2(t) = e^{-\lambda t} - \tfrac{1}{4}(\gamma + \lambda)^2.$$

Equation (14a) then becomes

$$\ddot{u} + (\gamma + \lambda)\dot{u} + e^{-\lambda t}u = e^{-2\lambda t}u^{-3}.$$

One can solve this equation for sufficiently large time under the assumption that u remains finite. We find that

$$u(t) = A + B(\gamma + \lambda)^{-1} \exp[-(\gamma + \lambda)t],$$

and the same result holds for the displacement $x(t)$. Since the increasing mass will cause the kinetic energy to vanish, the asymptotic energy will be purely potential and will depend on the final constant values of x and u.

Typical results are shown in figure 2. We see that, irrespectively of the value of λ, the fluctuation χ remains almost identical to the adiabatic value $\hbar/(2m(t)\Omega(t))$. It is insensitive to changes in the strength of the damping parameter when λ is small enough ($\lambda = 0{\cdot}1$ in units of the unperturbed frequency). For larger λ, a slight deviation from the adiabatic trend is observed according to different γ's. In contrast, damping has large effects on the evolution of the coordinate and momentum: this is reflected in the energy curves.

We observe that for large damping ($\gamma = 1$) it takes half a period to dissipate most of the initial energy, while for small $\gamma(\gamma = 0{\cdot}1)$, dissipation is slower and presents smooth oscillations. Both regimes converge asymptotically towards the adiabatic trend $E = 0{\cdot}5\hbar\omega(t)$.

5.1.2. Singularity at finite time. To illustrate the case in which the mass becomes infinite at a finite time we choose the representation

$$m = 1, \qquad t < 0$$
$$m = \exp[t/(T_1 - t)], \qquad t \geqslant 0.$$

T_1 is a parameter that fixes both the position of the singularity and the rate of increase of the mass. The calculations are presented in figure 3. The striking result is the fact that the final energy presents an inversion as a function of the damping parameter γ. As in the preceding case, the final energy is purely potential. For values of

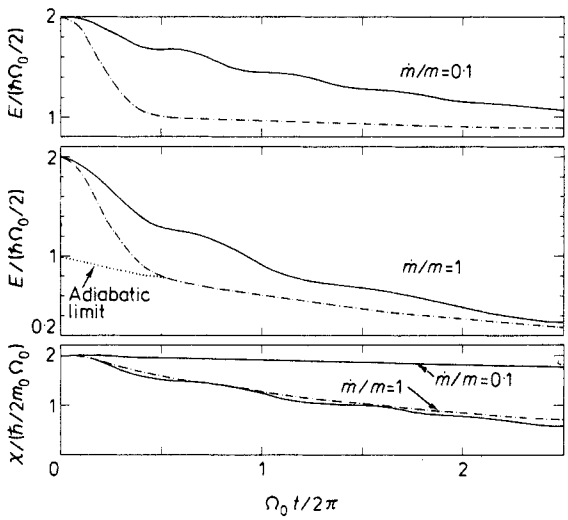

Figure 2. The same as figure 1, but for an exponentially increasing mass. Full lines correspond to the damping parameter $\gamma = 0{\cdot}1$, dashed-dotted lines correspond to $\gamma = 1$. Both lines coincide in the time evolution of χ when $\dot{m}/m = 0{\cdot}1$. The initial conditions are those of a displaced wave packet with the ground-state width.

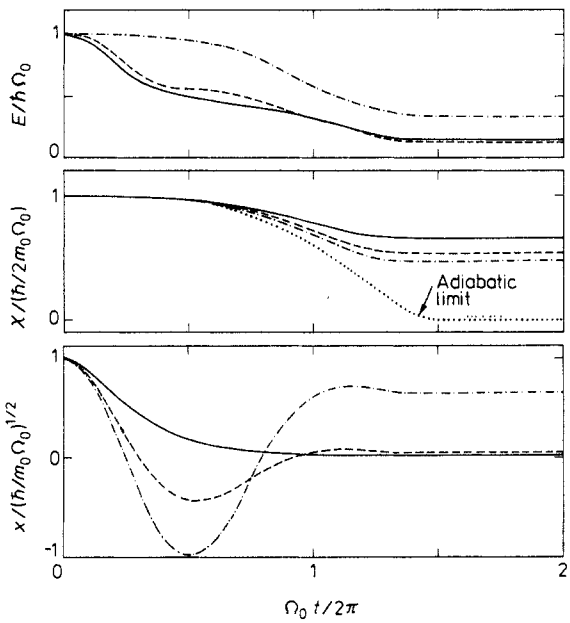

Figure 3. The same as figure 1, but with a mass going to infinity at $t = 2$. The full, dashed and dashed-dotted lines correspond respectively to $\gamma = 1{\cdot}99$, $0{\cdot}5$ and 0. The initial conditions are the same as in figure 2.

γ close to the critical one (i.e. $\gamma = 2\Omega$), the fluctuation χ overcompensates the attenuation of the position x and causes a large potential energy. In addition, we notice that the final displacement is not zero for small damping. The actual value of this

displacement depends on the initial phase; accordingly we may expect the final energy to present some range of variations for small γ's.

In contrast with the first example, we see that the evolution of χ presents a large departure from the adiabatic trend. During the process, there is an interplay between the role of damping and mass variation. For short times when $\dot{m}/m$ is small, the damping parameter γ determines the slope with which χ comes apart from the adiabatic curve. Afterwards, the term $\dot{m}/m$ overcomes γ and accounts for the final state of the system.

5.1.3. Periodic mass variation. A variable mass provides a means of simulating an input or removal of energy into the system. An explicit representation that accounts for several interesting features is a periodic perturbation on a constant mass, i.e.

$$m = 1, \qquad t < 0$$

$$m = 1 + \alpha \sin \lambda t, \qquad t \geqslant 0$$

with $|\alpha| < 1$.

This time variation in the mass can be traced to an external oscillating field; consequently, we can expect some resonant behaviour. A search through a wide range of parameters α and λ allowed us to obtain the results displayed in figure 4. In this case, the frequency of the mass is twice that of the unperturbed oscillator. The first point to notice is that both the energy and the fluctuation oscillate with the frequency of the mass; the deformation in the peaks of the energy curve can be associated with the fact that the position x is oscillating with the shifted frequency of the damped oscillator. Second, the resonant behaviour disappears when the strength of the damping becomes

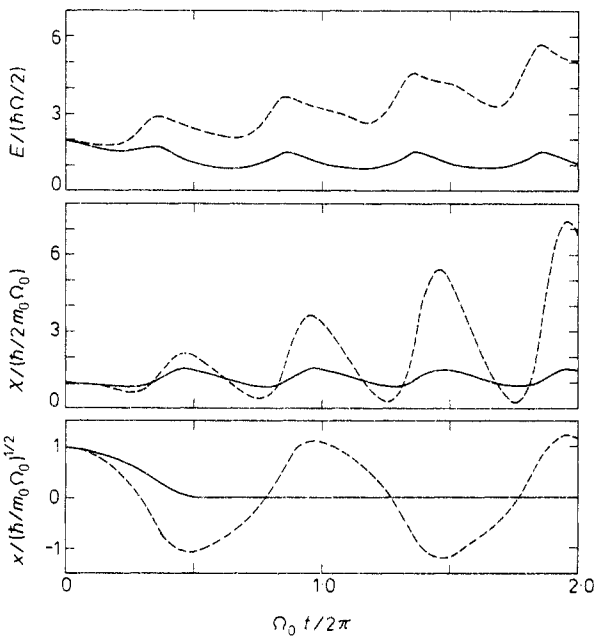

Figure 4. The same as figure 3, but with a mass with a periodic perturbation $m = 1 + 0{\cdot}5 \sin(2\Omega t)$. The dashed and full lines correspond to $\gamma = 0{\cdot}25$ and $\gamma = 1{\cdot}99$ respectively. The initial conditions are the same as in figure 2.

close to the critical value. In this case, χ and E perform constant amplitude oscillations around the ground-state values. A selection of the parameters α and λ, other than those corresponding to figure 4, yields modulation of the displayed curves and a much smoother increase of the amplitudes.

5.2. Infinite mass system at $t = 0$

It has been suggested (Myers 1979, private communication, Berlanger *et al* 1979) that the charge equilibration process during heavy-ion reactions may be pictured as the relaxation of a collective coordinate placed in a quadratic potential. As this equilibration is impossible when the ions are far apart (before and after the reaction), this suggests that the collective coordinate corresponds to an oscillator whose variable mass is infinite at $t = 0$, reaches a finite value during the interaction time and becomes infinite again when the two ions split apart. Not wanting to go into the physical details, we are going to take a simplified representation of the above mentioned situation; namely, an oscillator with constant stiffness $c = 1$, a constant damping γ and a mass given by the law

$$m(t) = \begin{cases} \exp(T_1/t - 1)^2 & \text{if } 0 \leqslant t < T_1 \\ 1 & \text{if } T_1 \leqslant t < T_1 + T_2 \\ \exp\left(\dfrac{T_3}{T_1 + T_2 + T_3 - t} - 1\right)^2 & \text{if } T_1 + T_2 \leqslant t \leqslant T_1 + T_2 + T_3. \end{cases}$$

A typical pattern for $m(t)$ is presented in figure 5 for a particular selection of the intervals T_1, T_2 and T_3.

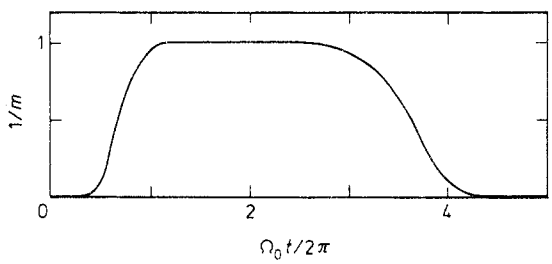

Figure 5. Typical evolution of the inverse of a mass that goes to infinity at $t = 0$ and $t = 5$ (see § 5·2 in text). The time intervals are $T_1 = 1$ and $T_2 = T_3 = 2$.

When $m(t)$ is very large, we can simplify the equations of motion for both first and second moments (equations (1) and (10)) provided that the initial p is finite and the initial χ is not zero.

In that case, we have

$$x = x_0, \qquad p = p_0 - (x_0/\gamma)[1 - \exp(-\gamma t)],$$
$$\chi = \chi_0, \qquad \phi = \hbar^2/4\chi_0 + (\chi_0/\gamma^2)[1 - \exp(-\gamma t)]^2,$$
$$\sigma = (\chi_0/\gamma)[1 - \exp(-\gamma t)], \qquad t \simeq 0.$$

These are the equations governing the initial motion when the initial correlation σ is zero, and it is interesting to realise that they are independent of the actual value of the

mass. A similar feature is expected in the vicinity of the second singular point, $t = T_1 + T_2 + T_3$.

In figure 6 we display the time dependence of the energy for various damping parameters and typical values of the time intervals. In that case, the equation governing the evolution of $\dot{E}$ is

$$\dot{E} = -(\gamma + \dot{m}/2m)(\phi + p^2)/m + \hbar^2\gamma/(4\chi m).$$

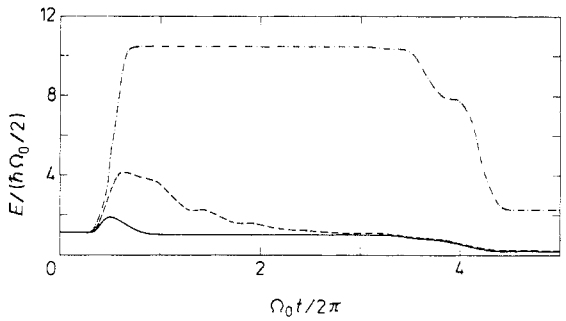

Figure 6. Time evolution of the total energy for the oscillation whose mass evolution is displayed in figure 5. The full, dashed and dashed-dotted lines correspond to damping parameters $\gamma = 1{\cdot}99$, $0{\cdot}25$, 0, respectively. The initial values are $\chi_0 = 0{\cdot}2$ and $x_0 = 1$.

For this particular mass law, $\dot{E} = 0$ for both $t = 0$ and $t = T_1 + T_2 + T_3$. This feature is clearly visible in figure 6.

Due to the action of the damping on ϕ and p^2, dramatic effects are induced on the rate of energy change even when $\dot{m}/m$ is large: the height of the plateau is strongly dependent on the value of γ; in addition, the larger the γ parameter, the shorter is the elapsed time before reaching this plateau.

As in a preceding example (§ 5.1.2), the final energy is only potential and depends upon the final value of fluctuations and displacements. In figure 6, we see that when $\gamma = 0$, there is a larger residual displacement that is determined by the history of the system (initial displacement, length of the plateau, etc.). A sensible amount of damping on a sufficiently long plateau causes x to vanish; accordingly, the final energy is given by the fluctuations only. These are displayed in figure 7. We have assumed a small initial width in order to represent a well-localised wave packet; this actual value is critical for determining the amplitude of the oscillations in χ. The wide oscillations of the

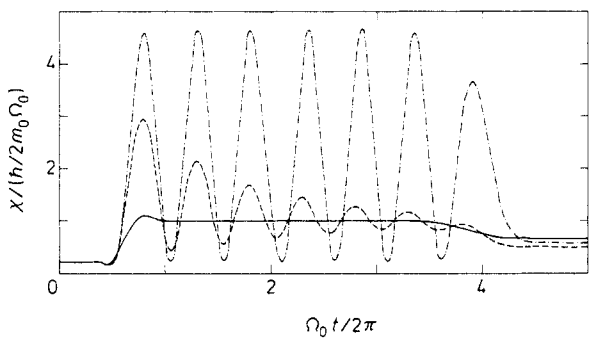

Figure 7. Time evolution of the fluctuation χ corresponding to the situation of figure 6.

undamped case are smoothed away by finite γ values and disappear completely when γ is close to the critical damping.

For sufficiently (not too) large values of damping, the system loses the memory of its past history in a couple of periods. In that case, the whole problem reduces to the study of the evolution of an initial ground state when a perturbation in the form of an increasing mass is superimposed.

For smaller amounts of damping, the final results are much more sensitive to the particular selection of the time intervals and initial conditions.

For the sake of completeness, in figure 8 we display the time evolution of the coordinate x. It is worthwhile to remark that although the frequency of the oscillations in the fluctuation is twice as large as that of the coordinate, both χ and x follow essentially the same pattern.

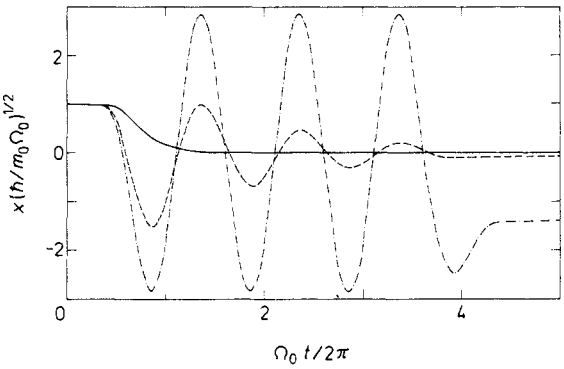

Figure 8. Time evolution of the position x corresponding to the situation of figure 6.

6. Conclusions

We have generalised both the existing procedures for the study of a classical, undamped, time-dependent oscillator and the methods for describing a quantal, damped, time-independent oscillator. We have been able to provide tools for the analysis of a general quantal, damped, time-dependent oscillator in the case in which the quantal state can be represented by a Gaussian wave packet.

We have found criteria to select, among those that are available to us, the only correct description for damping of a Gaussian wave packet in a quadratic field. This description is correct in the sense that it both preserves the uncertainty principle and yields the expected asymptotic behaviour. We have shown that this formalism holds even in the case of a mass reaching an infinite value. Kostin's frictional potential provides the right decay of any Gaussian wave packet to the ground state, in the presence of dissipation. We have illustrated possible applications of this method in a number of examples. One particular case that deserved our attention was the time evolution of the ground state when a perturbation, i.e. a displacement and a time-variation in the mass, is applied. There is a definite interplay between damping and mass variation; some important features are: (a) When the logarithmic derivative $\dot{m}/m$ is smaller than unity, the fluctuation χ remains close to the adiabatic limit $\hbar/2m(t)\Omega(t)$ and damping induces second-order effects in its evolution. In this situation, damping is mainly used to dissipate the energy of the motion. (b) When $\dot{m}/m$ becomes large (for

example in the vicinity of a singularity at a finite time) strong deviations from this adiabatic trend can be observed. Nevertheless, the initial slope of the departure from the adiabatic trend is determined by the value of the damping parameter. We have also seen that a perturbation in the ground state can account for a resonant behaviour; in this case the presence of dissipation provides attenuations of the motion and of the increase in the amplitude of the fluctuation χ; the critical damping ensures steady oscillations in both the fluctuation and the energy.

An interesting problem connected with the excitation of collective modes in some nuclear (i.e. heavy ion) reactions is that of an initially infinite mass, that lowers to a finite constant value and rises to infinity afterwards (§ 5.2). Such a situation involves a number of parameters, namely, initial width and displacement of the wave packet, duration of the mass decrease, length of the plateau, duration of the mass increase, and damping. Only a detailed treatment of the physics here contained can fix their actual values. However, we have illustrated the expected behaviour of the wave packet for a particular selection of the parameters, and shown that damping is essential in determining the evolution pattern.

As a final statement, we should remark that since there is no unique quantal description of friction, any particular application of dissipation models should be preceded by a critical examination, as we have intended here for the case of a Gaussian wave packet in the most general harmonic field.

Acknowledgments

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References

Albrecht K 1975 *Phys. Lett.* B**56** 127
Berlanger M, Gobbi A, Hanappe F, Lynen H, Ngô C, Olmi A, Sann H, Stelzer H, Richel H and Rivet M F 1979 *Z. Phys.* A**291** 133
Hasse W 1975 *J. Math. Phys.* **16** 2005
—— 1978 *J. Phys. A: Math. Gen.* **11** 1245
Hernandez E S and Remaud B 1980 *Phys. Lett.* to be published
Howard J E 1970 *Phys. Fluids* **13** 2407
Immele J D, Kan K K and Griffiin J J 1975 *Nucl. Phys.* A **241** 47
Kan K K and Griffin J J 1974 *Phys. Lett.* B **50** 241
Kanaï E 1948 *Prog. Theor. Phys.* **3** 440
Kostin M D 1975 *J. Statist. Phys.* **12** 145
Lewis H R 1967 *Phys. Rev. Lett.* **18** 510
Remaud B and Hernandez S E 1979 *Lawrence Berkeley Laboratory Report* No 9506
Senitzky I R 1960 *Phys. Rev.* **119** 670
Symon K R 1970 *J. Math. Phys.* **11** 1320